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Approximation by Entire Functions on Unbounded Domains in $\mathbb{C}^n$

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Given a function analytic in an unbounded domain of $\mathbb{C}^n$ with certain estimates of growth, we construct an entire function approximating it with a certain rate in some inner domain and give estimates of growth of the approximating function. This extends well-known results of M. V. Keldysh to several variables and more general domains.

Let Ω be a set in the complex plane $\mathbb{C}$ and let $f(z)$ be a continuous function on Ω holomorphic in the interior of Ω. Well-known results of Keldysh, Arakelyan, and other authors describe the conditions under which uniform or tangent approximation by entire functions is possible. We refer to [1, 2, 5] for the exposition of results and more complete references.

We are interested in the problems of control of growth of the approximating function. Normally, such a control is possible if we approximate the function $f(z)$ on some set lying inside Ω. About 40 years ago M. V. Keldysh proved a number of results of the following kind. Given a function analytic in an angle (in a strip) with certain estimates of growth, it is possible to construct an entire function which approximates the given one in some interior angle (strip) with a certain rate and to estimate its order of growth in terms of the order of growth of the given function, characteristics of the angle (strip), and the rate of approximation. Keldysh's proofs seem rather complicated. The aim of the present paper is to suggest a different (and more simple) method to prove results of this type and to extend these results to several variables and more general (unbounded) domains.

This becomes possible due to application of Hörmander's $\bar{\partial}$-methods. Being essentially "multidimensional," these tools nonetheless often bring new results in one complex variable and give rather clear proofs. It is enough to mention the recent short and elegant proof of Arakelyan's approximation theorem [6]. Here we try to treat the problems in a similar way.

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Throughout the paper $\omega(z)$ and $\phi(z)$ are plurisubharmonic functions in $\mathbb{C}^n$, both possessing the "non-oscillating" property

$$(u)^{[1]}(z) \leqslant -A(-u)^{[1]}(z) + B,$$

where by $u^{[r]}(z)$ we denote $\sup\{u(w) : |z-w| < r\}$. Assume also that $\phi(z) \geqslant 0$ *and* $\log(1+|z|) = o(\phi(z))$, $|z| \to \infty$. For $\varepsilon \geqslant 0$ we denote by Ω_ε the set

$$\Omega_\varepsilon = \{z \in \mathbb{C}^n : \omega(z) < -\varepsilon\phi(z)\}$$

and suppose that

$$\forall \varepsilon_1 > \varepsilon_2 : \inf\{|z_1 - z_2| : z_1 \in \Omega_{\varepsilon_1}, z_2 \in \mathbb{C}^n \backslash \Omega_{\varepsilon_2}\} \stackrel{\text{def}}{=} r_{\varepsilon_1, \varepsilon_2} > 0, \tag{1}$$

which is a kind of smoothness condition on ω and ϕ.

THEOREM 1. *Let $f(z)$ be an analytic function in Ω_0 satisfying the estimate*

$$|f(z)| \leqslant C_f e^{C_f \phi(z)}, \qquad z \in \Omega_0.$$

Then for each $\varepsilon > 0$ and each $N \geqslant 1$ there exists an entire function $g(z)$ such that

$$|f(z) - g(z)| \leqslant Ce^{-N\phi(z)}, \qquad z \in \Omega_\varepsilon, \tag{2}$$

$$|g(z)| \leqslant Ce^{C\max(N, C_f)\cdot((2/\varepsilon)\cdot\omega + \phi)(z)}, \qquad z \in \mathbb{C}^n, \tag{3}$$

where C does not depend on N.

As we have mentioned already, our main tool is the well-known result of Hörmander [3]. Here is the version we need.

THEOREM H. *Let α be a $(0, 1)$-form in $\mathbb{C}^n$ with $\bar{\partial}\alpha = 0$ and*

$$\|\alpha\|_\psi^2 \stackrel{\text{def}}{=} \int_{\mathbb{C}^n} |\alpha|^2 e^{-\psi}\, d\lambda < \infty$$

for some plurisubharmonic function ψ. Then there exists a solution β of the equation $\bar{\partial}\beta = \alpha$ such that

$$\|\beta\|_{\tilde{\psi}}^2 \leqslant \tfrac{1}{2}\|\alpha\|_\psi^2$$

with $\tilde{\psi} = \psi + 2\log(1+|z|^2)$.

Proof of Theorem 1. Set $\omega_1 = \omega + (\varepsilon/2)\phi(z)$. Let $\chi(z) \in C^\infty(\mathbb{C}^n)$ be a function with the properties $\chi(z) = 1$ if $z \in \overline{\Omega_{\varepsilon/4}}$, $\chi(z) = 0$ if $z \in \mathbb{C}^n \backslash \Omega_0$, $0 \leqslant \chi(z) \leqslant 1$, $\forall z$. In view of (1), $\chi(z)$ may be chosen so that $|\bar{\partial}\chi(z)| \leqslant C_1$. We construct the approximating function $g(z)$ in the form

$$g(z) = \chi(z) f(z) - \beta(z),$$

where

$$\bar{\partial}\beta(z) = \bar{\partial}\chi(z) \cdot f(z) \overset{\text{def}}{=} \alpha(z).$$

One can easily see that $|\alpha(z)| \leqslant C_1 C_f e^{C_f \phi(z)}$. Let $M = (8/\varepsilon)\max(N, C_f)$. We have

$$\int_{\mathbb{C}^n} |\alpha(z)|^2 e^{-M\omega_1(z)} \frac{d\lambda}{(1+|z|^2)^{n+1}}$$
$$\leqslant C_1^2 C_f^2 \int_{\Omega_0 \setminus \Omega_{\varepsilon/4}} e^{2C_f\phi(z)} e^{-M\omega_1(z)} \frac{d\lambda}{(1+|z|^2)^{n+1}}.$$

Since

$$\omega_1(z) \geqslant -\frac{\varepsilon}{4}\phi(z) + \frac{\varepsilon}{2}\phi(z) = \frac{\varepsilon}{4}\phi(z) \qquad \text{for} \quad z \in \Omega_0 \setminus \Omega_{\varepsilon/4},$$

we have $M\omega_1(z) \geqslant \max(2N, 2C_f)\,\phi(z)$. Hence

$$\int_{\mathbb{C}^n} |\alpha(z)|^2 e^{-M\omega_1(z)} \frac{d\lambda}{(1+|z|^2)^{n+1}} \leqslant C_1^2 C_f^2 \int_{\Omega_0 \setminus \Omega_{\varepsilon/4}} \frac{d\lambda}{(1+|z|^2)^{n+1}}$$
$$\leqslant C_1^2 C_f^2 \int_{\mathbb{C}^n} \frac{d\lambda}{(1+|z|^2)^{n+1}} < \infty.$$

Note that the right-hand side of the estimate does not depend on N.

By Theorem H (with $\psi(z) = M\omega_1(z) + (n+1)\log(1+|z|^2)$, which is clearly plurisubharmonic), the equation $\bar{\partial}\beta = \alpha$ has a solution β such that

$$\int_{\mathbb{C}^n} |\beta(z)|^2 e^{-M\omega_1(z)} \frac{d\lambda}{(1+|z|^2)^{n+3}} \leqslant \frac{1}{2} C_1^2 C_f^2 \int_{\mathbb{C}^n} \frac{d\lambda}{(1+|z|^2)^{n+1}} < \infty. \quad (4)$$

We estimate $|\beta(z)|$ first for $z \in \Omega_{3\varepsilon/8}$. Set $R_1 = \min(1, r_{3\varepsilon/8, \varepsilon/4}, r_{\varepsilon, 3\varepsilon/4})$. Since $\bar{\partial}\beta = 0$ in $\Omega_{\varepsilon/4}$, i.e., β is holomorphic, and the ball $B(z, R_1)$ for $z \in \Omega_{3\varepsilon/8}$ is contained in $\Omega_{\varepsilon/4}$, from the integral estimate (4) we get (τ_n denotes the volume of the unit ball in $\mathbb{C}^n$)

$$|\beta(z)|^2 \leqslant \frac{1}{\tau_n R_1^{2n}} \int_{B(z R_1)} |\beta(\zeta)|^2 \, d\lambda$$
$$\leqslant C_2 e^{M\omega_1^{[R_1]}(z)} (1+|z|^2)^{n+3} \int_{B(z R_1)} |\beta(\zeta)|^2 e^{-M\omega_1(\zeta)} \frac{d\lambda}{(1+|\zeta|^2)^{n+3}}$$
$$\leqslant \frac{1}{2} C_2 C_1^2 C_f^2 (1+|z|^2)^{n+3} e^{M\omega_1^{[R_1]}(z)} \int_{\mathbb{C}^n} \frac{d\lambda}{(1+|\zeta|^2)^{n+3}}, \qquad z \in \Omega_{3\varepsilon/8},$$

that is,

$$|\beta(z)| \leqslant C_3 e^{M\omega_1^{[R_1]}(z)/2}(1+|z|^2)^{(n+3)/2}, \qquad z \in \Omega_{3\varepsilon/8}, \tag{5}$$

where C_3 does not depend on N.

For $z \in \Omega_\varepsilon$ we have $|f(z)-g(z)| = |\beta(z)|$. Taking into account that $R_1 \leqslant r_{\varepsilon, 3\varepsilon/4}$ and that $\omega_1(z) \leqslant -(\varepsilon/4)\,\phi(z)$ for $z \in \Omega_{3\varepsilon/4}$, we get from (5)

$$\begin{aligned}|f(z)-g(z)| &\leqslant C_3(1+|z|)^{n+3}\, e^{(M/2)\cdot(-(\varepsilon/4)\,\phi)^{[R_1]}(z)} \\ &\leqslant C_3(1+|z|)^{n+3}\, e^{\max(N, C_f)(-\phi)^{[1]}(z)}, \qquad z \in \Omega_\varepsilon.\end{aligned}$$

Since we are able to replace N by KN with $K>1$ not depending on N, so that the term $(1+|z|)^{n+3}$ is "swallowed" by $\exp(-KN\phi)$, and in view of the "non-oscillation" property we conclude that (2) holds.

We estimate $|g(z)|$ now. For $z \in \Omega_{3\varepsilon/8}$, in view of (5) and since $\omega_1(\zeta) \leqslant (-\varepsilon/4+\varepsilon/2)\,\phi(\zeta) = (\varepsilon/4)\,\phi(\zeta)$ for $\zeta \in B(z, R_1)$, we have

$$\begin{aligned}|g(z)| &\leqslant |f(z)| + |\beta(z)| \leqslant C_f e^{C_f\phi(z)} + C_3 e^{M\omega_1^{[1]}(z)/2} \\ &\leqslant C_f e^{C_f\phi(z)} + C_3 e^{M(\varepsilon/8)\,\phi^{[1]}(z)} \leqslant C_4 e^{\max(N, C_f)\,\phi^{[1]}(z)}.\end{aligned}$$

For $z \in \mathbb{C}^n \setminus \Omega_{3\varepsilon/8}$ the estimate is slightly different:

$$\begin{aligned}|g(z)|^2 &\leqslant \frac{1}{\tau_n} \int_{B(z,1)} |g(\zeta)|^2\, d\lambda \\ &\leqslant \frac{2}{\tau_n}\left(\int_{B(z,1)} |\chi(\zeta)|^2\, |f(\zeta)|^2\, d\lambda + \int_{B(z,1)} |\beta(\zeta)|^2\, d\lambda\right) \\ &\leqslant 2C_f^2 e^{2C_f\phi^{[1]}(z)} + C_5 e^{M\omega_1^{[1]}(z)}(1+|z|^2)^{n+3} \\ &\quad \times \int_{B(z,1)} |\beta|^2 e^{-M\omega_1} \frac{d\lambda}{(1+|\zeta|^2)^{n+3}} \\ &\leqslant 2C_f^2 e^{2C_f\phi^{[1]}(z)} + C_5 e^{M\omega_1^{[1]}(z)}(1+|z|^2)^{n+3} \\ &\quad \times \int_{\mathbb{C}^n} |\beta|^2 e^{-M\omega_1} \frac{d\lambda}{(1+|\zeta|^2)^{n+3}}.\end{aligned}$$

This estimate, together with the previous one and the "non-oscillation" condition, implies (3). The theorem is proved.

It is easy to obtain results of Keldysh type as corollaries. To show this denote by W_α the angle $\{z \in \mathbb{C}\colon |\arg z| < \alpha/2\}$.

THEOREM 2. *Let $\rho=\pi/(2\pi-\alpha)$, $\alpha\leqslant\pi$, and let f be an analytic function in W_α, $|f(z)|\leqslant Ke^{K|z|^\rho}$. Then $\forall\eta,\gamma>0$, $\forall\delta\in(0,\alpha)$ there exists an entire function $g(z)$, $|g(z)|\leqslant Ce^{C|z|^\rho}$, such that*

$$|f(z)-g(z)|<\eta e^{-|z|^\rho},\qquad z\in W_{\alpha-\delta}\cap\{|z|\geqslant\gamma\}.$$

Remark. If the function $f(z)$ is analytic in some neighborhood of the origin, it is possible to modify $g(z)$ so that the approximation takes place near the origin as well, and we get exactly Keldysh's result (see, e.g., [2, Chap. IV, B, Theorem 2]).

Proof of Theorem 2. Note that $\rho\in(0.5,1]$. We set $n=1$ and define $\omega(z)=|z|^\rho\cos\rho(\arg z-\pi)$ ($\arg z$ is taken from $[0,2\pi]$), $\phi(z)=\max\{|z|^\rho,\gamma^\rho\}$. In the notations introduced above $\Omega_0=W_\alpha$ and the set Ω_ε for $\varepsilon<|\cos\rho\pi|$ contains the angle $W_{\alpha-\beta}$ (with $\beta=\beta(\varepsilon)=(2/\rho)\arccos(-\varepsilon)-\pi/\rho$) without some neighborhood of the origin ($|z|>\gamma/2$).

For such Ω_ε the condition (1) takes place. Choose ε so that $\beta(\varepsilon)<\delta$. Then $\Omega_\varepsilon\supset W_{\alpha-\delta}\cap\{|z|\geqslant\gamma\}$. Taking N large enough that $Ce^{-N\max\{|z|^\rho,\gamma^\rho\}}<\eta e^{-|z|^\rho}$ in $W_{\alpha-\delta}\cap\{|z|\geqslant\gamma\}$, and applying Theorem 1, we get the required assertion.

Now denote by Π_h the strip $\{z\in\mathbb{C}\colon|\mathrm{Re}\,z|<h\}$.

THEOREM 3. *Let $\rho=\pi/2h$ and let f be analytic in Π_h, $\log|f(z)|\leqslant Ke^{\rho|\mathrm{Im}\,z|}$. Then $\forall\eta>0$, $\forall\delta\in(0,h)$, there exists an entire function $g(z)$, $\log|g(z)|\leqslant Ce^{\rho|\mathrm{Im}\,z|}$, such that*

$$|f(z)-g(z)|<\eta e^{-e^{\rho|\mathrm{Im}\,z|}},\qquad z\in\Pi_{h-\delta}.$$

Remark. The result may be found in [4, 5]. As is easily seen from the proof below, we can prove more, namely, $f(z)$ can be defined on an infinite number of strips situated periodically, and the approximation is simultaneous on all interior strips.

Proof of Theorem 3. Set $n=1$, $\omega(z)=-\cosh\rho\,\mathrm{Im}\,z\cdot\cos\rho\,\mathrm{Re}\,z$, $\phi(z)=\cosh\rho\,\mathrm{Im}\,z$. Then $\Omega_0=\tilde\Pi_h$, $\Omega_\varepsilon=\tilde\Pi_{h-(1/\rho)\arccos\varepsilon}$, where we denote by $\tilde\Pi$ the set

$$\bigcup_{k=-\infty}^{+\infty}\left\{\zeta=z+k\frac{2\pi}{\rho},\,z\in\Pi\right\}.$$

We define $f(z)$ to be zero in those components of $\tilde\Pi$ where it is not yet defined. Applying Theorem 1 with ε small enough and N large enough, we get the assertion we need.

Results of slightly different type may be obtained if we take, for instance, $\omega=\log|F|+C|z|^\rho$, $C>0$, F being an entire function of finite type with respect to order ρ, and $\phi = |z|^{\rho}$. We do not formulate the corresponding statement here and do not concentrate on other domains in $\mathbb{C}$, for which appropriate functions ω and ϕ may be chosen.

Let us look at the situation in several variables. If we are interested in extending Keldysh's result concerning an angle, we have a rich choice of cones in $\mathbb{C}^n$ corresponding to choices of ω and ϕ. Below we mention without proof just one possible result.

For example, take a round cone

$$C_\alpha = \{z \in \mathbb{C}^2 \colon (\operatorname{Re} z_1)^2 + (\operatorname{Im} z_1)^2 + (\operatorname{Im} z_2)^2 \leqslant \alpha (\operatorname{Re} z_1)^2\}, \qquad \alpha < 1.$$

THEOREM 4. *Let N be a positive number. Each function analytic in the cone C_α for some $\alpha \in (0, 1)$ and having order not greater than 2 may be approximated in each interior cone C_β, $\beta \in (0, \alpha)$ without some neighborhood of the origin, by an entire function of order not greater than 2 with the rate* $\exp(-N\,|z|^2)$.

Remark. One feels that two (which does not depend on α) is not the least possible order for the approximation in C_α. Still we do not know how to choose a better ω in this case.

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